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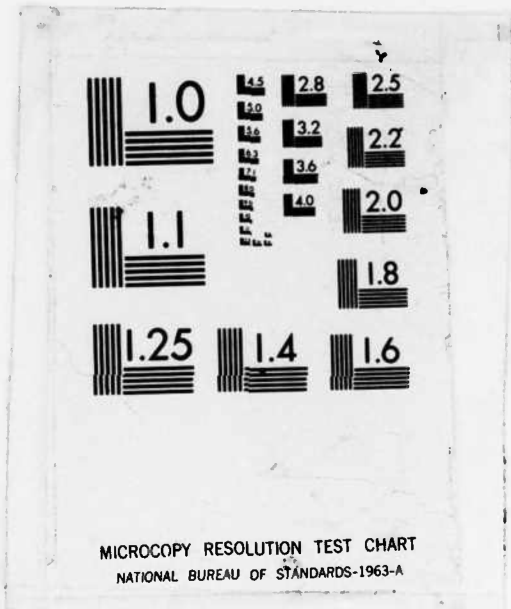
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FACETS OF THE THREE-INDEX ASSIGNMENT POLYTOPE

by

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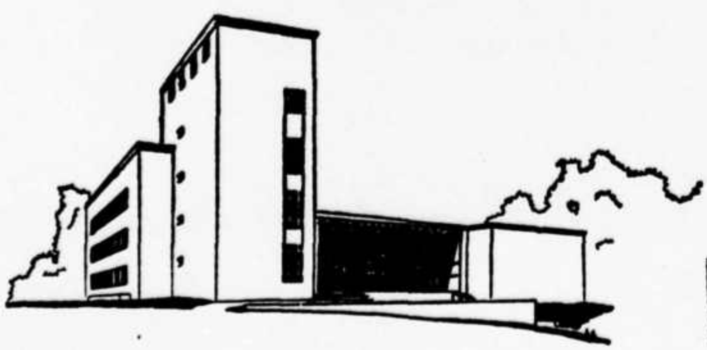

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Abstract

Given three disjoint n-sets and the family of all weighted triplets that contain exactly one element of each set, the 3-index assignment (or 3-dimensional matching) problem asks for a minimum-weight subcollection of triplets that covers exactly (i.e., partitions) the union of the three sets. Unlike the common (2-index) assignment problem, the 3-index problem is NP-complete. ~~In~~ this paper ~~we~~ examines the facial structure of the 3-index assignment polytope (the convex hull of feasible solutions to the problem) with the aid of the intersection graph of the coefficient matrix of the problem's constraint set. In particular, we describe the cliques of the intersection graph as belonging to three distinct classes, and show that cliques in two of the three classes induce inequalities that define facets of our polytope. Furthermore, we give an $O(n^4)$ procedure (to the 4th power) (note that the number of variables is n^3 (cubed)) for finding a facet-defining clique-inequality violated by a given noninteger solution to the linear programming relaxation of the 3-index assignment problem, or showing that no such inequality exists. We then describe the odd holes of the intersection graph and identify two classes of facets associated with odd holes that are easy to generate. One class has coefficients of 0 or 1, the other class coefficients of 0, 1 or 2. No odd hole inequality has left hand side coefficients greater than two.

1. Introduction

The *(axial) three-index assignment problem*, to be denoted AP3, also known as the *three-dimensional matching problem*, can be stated as follows: given three disjoint n-sets, I, J, and K, and a weight c_{ijk} associated with each ordered triplet $(i,j,k) \in I \times J \times K$, find a minimum-weight collection of n disjoint triplets $(i,j,k) \in I \times J \times K$. This problem is called *axial* to distinguish it from another three-index assignment problem, known as *planar*.

An alternative interpretation of AP3 is as follows. A graph is *complete* if all of its nodes are pairwise adjacent. A maximal complete subgraph of a graph is a *clique*. A graph is *k-partite* if its nodes can be partitioned into k subsets such that no two nodes in the same subset are joined by an edge. It is *complete k-partite*, if every node is adjacent to all other nodes except those in its own subset. The complete k-partite graph with n_i nodes in its i^{th} part (subset) is denoted $K_{n_1,n_2,\ldots,n_k}$.

Consider now the complete tri-partite graph $K_{n,n,n}$ with node set $R = I \cup J \cup K$, $|I| = |J| = |K| = n$. Figure 1 shows $K_{n,n,n}$ for n=2 and n=3. $K_{n,n,n}$ has 3n nodes and n^3 cliques, all of which are triangles containing exactly one node from each of the three sets I,J,K. Let (i,j,k) denote the clique induced by the node set {i,j,k}. If a weight c_{ijk} is associated with each clique (i,j,k), then AP3 is the problem of finding a minimum-weight exact clique cover of the nodes of $K_{n,n,n}$, where an exact clique cover is a set of cliques that partitions the node set R.

AP3 can be stated as a 0-1 programming problem as follows:

$$
\begin{aligned}
\min \quad & \textstyle\sum(c_{ijk}x_{ijk}: i \in I,\ j \in J,\ k \in K) \\
\text{s.t.} \quad & \textstyle\sum(x_{ijk}: j \in J,\ k \in K) = 1, \quad \forall\, i \in I \\
& \textstyle\sum(x_{ijk}: i \in I,\ k \in K) = 1, \quad \forall\, j \in J \\
& \textstyle\sum(x_{ijk}: i \in I,\ j \in J) = 1, \quad \forall\, k \in K \\
& \qquad x_{ijk} \in \{0,1\} \quad \forall\, i,j,k
\end{aligned}
$$

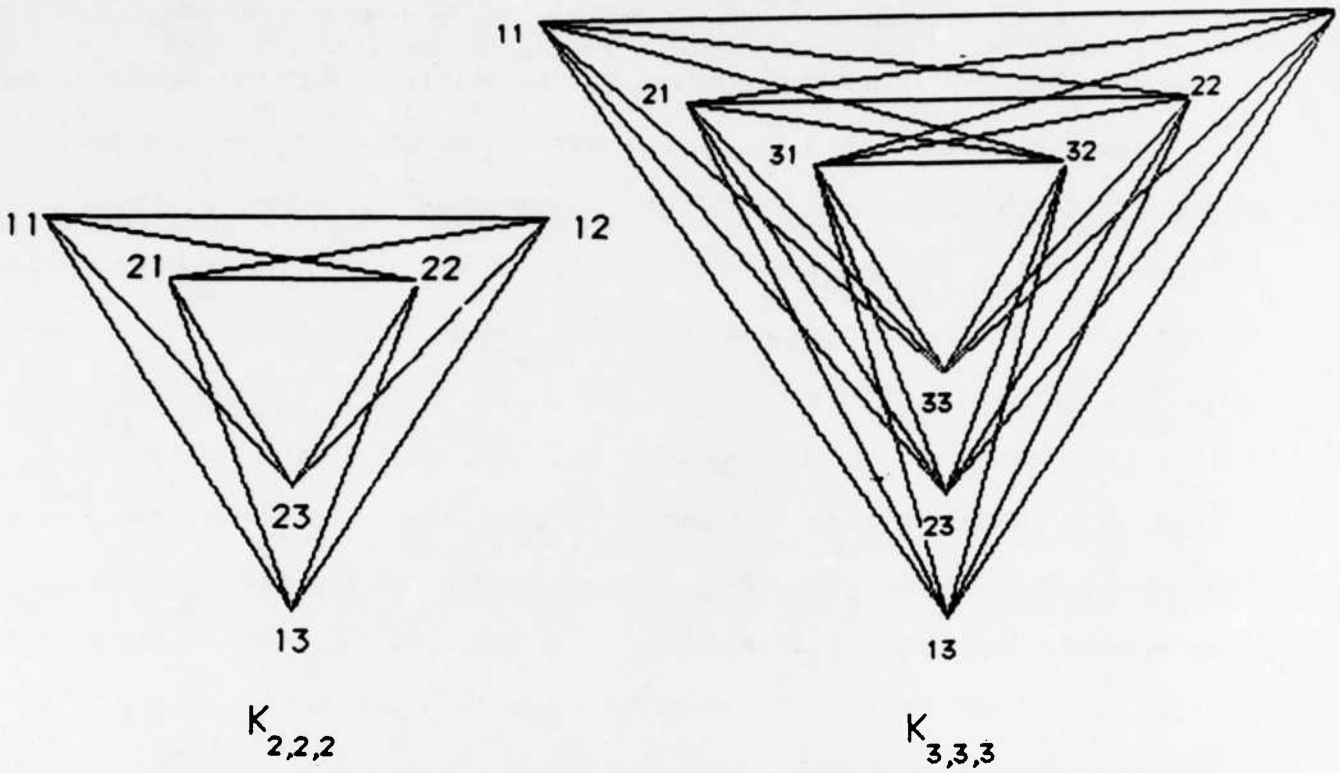


Figure 1

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$. The coefficient matrix of AP3 for the case n=3 is:

$$\left(\begin{array}{ccccccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&&&&&&&&&&&&&&&&&& \\
&&&&&&&&&1&1&1&1&1&1&1&1&1&&&&&&&&& \\
&&&&&&&&&&&&&&&&&&1&1&1&1&1&1&1&1&1 \\
1&1&1&&&&&&&1&1&1&&&&&&&1&1&1&&&&&& \\
&&&1&1&1&&&&&&&1&1&1&&&&&&&1&1&1&&& \\
&&&&&&1&1&1&&&&&&&1&1&1&&&&&&&1&1&1 \\
1&&&1&&&1&&&1&&&1&&&1&&&1&&&1&&&1&& \\
&1&&&1&&&1&&&1&&&1&&&1&&&1&&&1&&&1& \\
&&1&&&1&&&1&&&1&&&1&&&1&&&1&&&1&&&1
\end{array}\right)$$

We will denote by $AP3_n$ the (axial) 3-index assignment problem of ***order*** n (i.e., defined for n-sets), by A_n the coefficient matrix of its constraint set, and by I_n, J_n, K_n the 3 associated index sets. The row and column index sets of A_n will be denoted by R_n and S_n respectively. Clearly, $|R_n| = |I_n| + |J_n| + |K_n| = 3n$ and $S_n = |I_n| \times |J_n| \times |K_n| = n^3$.

In terms of $K_{n,n,n}$, A_n is the incidence matrix of nodes versus cliques (triangles): it has a row for every node and a column for every clique of $K_{n,n,n}$.

As usual, the ***support*** of a (row or column) vector is understood to mean the index set of its nonzero components. Each element of S (that indexes a column of A_n and a clique of $K_{n,n,n}$) will also be used to denote the support of the given column of A_n and the node set of the given clique (triangle) of $K_{n,n,n}$. Thus, if a^s has support (i,j,k) (i.e., if clique s of $K_{n,n,n}$ has node set {i,j,k}), we will write $s = (i,j,k)$ or $a^s = a^{ijk}$, meaning that column a^s has ones in positions $i \epsilon I$, $j \epsilon J$ and $k \epsilon K$.

AP3 is a close relative of the (axial) ***3-dimensional transportation problem***, in which the right hand sides of the constraints can be any positive integers, the sets I,J,K are not necessarily equal in size, and the integrality constraints are relaxed. This is in turn a generalization of the

well-known transportation problem, a special case of which is the simple assignment problem.

The 3-dimensional transportation problem (TR3) in these and other formulations was first studied by Schell [20]. The literature on this problem includes the references [2,5,9,10,12,13,14,15,18,19,20,21]. The original motivation for considering this model was a problem in the transportation of goods of several types from multiple sources to multiple destinations. Applications of AP3 mentioned in the literature include the following (Pierskalla [18,19]).

- In a rolling mill with $|I|$ soaking pits (temperature stabilizing baths), schedule $|K|$ ingots through the pits so as to minimize idle-time for the rolling mill (the next stage in the process).
- Find a minimum cost schedule of a set of capital investments (e.g., warehouses or plants) in different locations at different times.
- Assign troops to locations over time to maximize a measure of capability.
- Launch a number of satellites in different directions at different altitudes to optimize coverage or minimize cost.

AP3 is known to be an NP-complete problem [11]. Obviously, AP3 is a special case of the ***set partitioning problem***:

(SPP) $\max \{cx \mid Bx = e,\ x \in \{0,1\}^q\}$,

where B is a matrix of zeroes and ones and e is a vector of ones. A close relative of (SPP) is the ***set packing problem*** (SP), obtained from (1) by replacing $=$ with $\leq$.

For properties of (SPP) and (SP) see the survey [3].

Let P_I denote the convex hull of feasible solutions to $AP3_n$, i.e.,

$P_I = \mathrm{conv}\{x \in \{0,1\}^{n^3} \mid A_n x = e\}$

<u>Theorem 1.1</u>. P_I has $(n!)^2$ vertices.

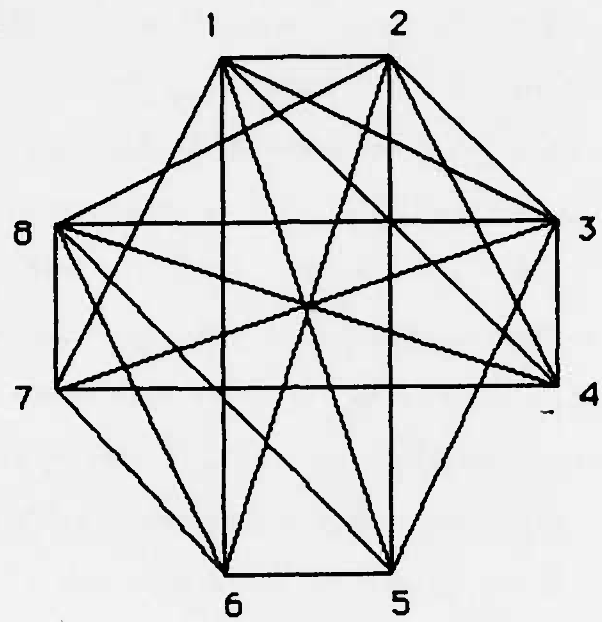
<u>Proof</u>. Let P_I^k denote the polyhedron P_I for $n = k$. For $n = 2$, the statement is true (by inspection). Suppose it is true for $n = 2,\ldots,r$, and let $n = r+1 \geq 3$. There are n^2 variables x_{njk} that have a nonzero coefficient in row n, and setting $x_{njk} = 1$ for any one of them defines a face of P_I^n which is precisely the polyhedron P_I^{n-1}. By hypothesis, P_I^{n-1} has $((n-1)!)^2$ vertices; hence P_I^n has $n^2 \times ((n-1)!)^2 = (n!)^2$ vertices.||

The *intersection graph* $G_A = (V,E)$ of a 0-1 matrix A has a node s for every column a^s of A, and an edge (s,t) for every pair of columns a^s, a^t such that $a^s \bullet a^t \neq 0$. The intersection graph G_{A_n} of A_n is the *clique-intersection graph* of $K_{n,n,n}$, i.e., G_{A_n} has a node for every clique (triangle) of $K_{n,n,n}$, and an edge for every pair of triangles that share some node of $K_{n,n,n}$. The graph G_{A_n} for $n=2$ is shown in figure 2.

Although the 3-index assignment problem has a sizeable literature, no work has been done until recently on describing the polytope P_I. In this paper we apply the tools of polyhedral combinatorics to $AP3_n$ and obtain a partial characterization of the facial structure of P_I. In particular, in section 2 we identify three classes of cliques of the intersection graph of A_n and show that they are exhaustive. These cliques are known to induce facets of the polytope

$$\tilde{P}_I = \text{conv}\{x \in \{0,1\}^{n^3} \mid A_n x \leq e\},$$

the set packing relaxation of the set partitioning polytope P_I. In section 3 we show that two of the 3 classes of cliques also induce facets of P_I, and that these facets are all distinct. In section 4 we give an $O(n^4)$ procedure for detecting a clique inequality violated by some solution to the linear programming relaxation of P_I, or showing that no such inequality exists. Section 5 describes the odd holes (odd-length chordless cycles) of the



G_{A_2}

Figure 2

intersection graph of A_n. Odd holes are known [17] to give rise to a class of facets of $\tilde{P}_I$, the set packing relaxation of P_I, and Euler [8a] has recently described a family of facets of P_I associated with the odd holes of maximum length, i.e., of length $2n - 1$. In section 6 we describe two classes of facets of P_I associated with odd holes of arbitrary length, one having left hand side coefficients of 0 or 1, the other one having coefficients of 0, 1 and 2. We also show that no odd hole inequality can have a left hand side coefficient greater than 2. An earlier version of our paper, containing sections 1-4, was circulated under [3a].

Since for $n = 1$ P_I reduces to a single point, we assume throughout the rest of the paper that $n \geq 2$.

2. The Cliques of G_A

In this section we identify all the cliques of G_A, the intersection graph of A.

For any subset $V \subseteq S$ of the node set of G_A, we will denote by $\langle V \rangle$ the subgraph induced by V. For $r \in R$, we will denote by S^r the support of row r of A, i.e., $S^r := \{s \in S | a_{rs} = 1\}$.

Proposition 2.1. For each $r \in R$, the node set S^r induces a clique (of cardinality n^2) in G_A.

Proof. The subgraph $\langle S^r \rangle$ is obviously complete. To see that it is maximal, assume w.l.o.g. that $r \in I$, and let $s \in S \backslash S^r$ be arbitrarily chosen, $s = (i_0, j_0, k_0)$. Since S^r contains all triplets whose first element is r, it contains a triplet $t \in S^r$, $t = (r,j,k)$, such that $r \neq i_0$, $j \neq j_0$, $k \neq k_0$. Hence $S^r \cup \{s\}$ does not induce a complete subgraph of G_A; and since this is true of any $s \in S \backslash S^r$, the subgraph of G_A induced by S^r is maximally complete, i.e., a clique. Furthermore, $|S^r| = n^2$ for all $r \in R$.||

The set of cliques defined by Proposition 2.1 will be called *class 1*. Clearly, the number of class 1 cliques is $3n$. In terms of $K_{n,n,n}$, the clique of class 1 corresponding to row r of A contains those nodes of the intersection graph G_A, whose associated triangles in $K_{n,n,n}$ share node r of $K_{n,n,n}$.

Proposition 2.2. For every $s \in S$, let

$$T(s) = \{t \in S \setminus \{s\} \mid a^s \bullet a^t = 2\}.$$

Then the node set $\{s\} \cup T(s)$ induces a clique of size $3n - 2$ in G_A.

Proof. Let $s = (i_0,j_0,k_0)$, and let $t_1,t_2 \in T(s)$ be chosen arbitrarily, with $t_1 \neq t_2$. Since each of t_1 and t_2 contains two of the three elements i_0,j_0,k_0, t_1 and t_2 must have at least one element in common. Hence the node set $\{s\} \cup T(s)$ induces a complete subgraph in G_A. Now let $u \in S \setminus (\{s\} \cup T(s))$. Then the triplet $u = \{i,j,k\}$ contains at most one element of s. If $a^u \bullet a^s = 0$, we are done. If $a^u \bullet a^s = 1$, assume w.l.o.g. that $i = i_0$; then $j \neq j_0$ and $k \neq k_0$. By definition, T(s) contains some $t = (i_*,j_0,k_0)$ such that $i_* \neq i_0(=i)$. But then $a^u \bullet a^t = 0$, i.e., $\{u\} \cup \{s\} \cup T(s)$ does not define a complete subgraph of G_A. Since the choice of u was arbitrary, the subgraph defined by $\{s\} \cup T(s)$ is maximal complete.

For each $s \in S$ and for each of the three pairs of the triplet $s = (i_0,j_0,k_0)$, there are $n-1$ other triplets in S containing the same pair; hence $|T(s)| = 3(n-1)$, and thus $\{s\} \cup T(s)$ has $3n-2$ elements.||

The set of cliques defined in Proposition 2.2 will be called *class 2*. There is exactly one clique of class 2 for every column of A, and there is no double counting; hence the number of class 2 cliques is n^3. In terms of $K_{n,n,n}$, the clique of class 2 corresponding to column $s = (i_0,j_0,k_0)$ of A contains the node of G_A corresponding to the clique (i_0,j_0,k_0) of $K_{n,n,n}$, along with the $3(n-1)$ nodes of G_A corresponding to those cliques of $K_{n,n,n}$ that share an edge (a pair of nodes) with the clique (i_0,j_0,k_0).

<u>Proposition 2.3</u>. For every ordered pair $s,t \in S$ such that $a^s \cdot a^t = 0$, let $t_1, t_2, t_3 \in S \setminus \{s,t\}$ be the (uniquely defined) triplets such that

$$a^s \bullet a^{t_i} = 1, \quad a^t \bullet a^{t_i} = 2, \qquad i = 1,2,3.$$

Then the node set $\{s,t_1,t_2,t_3\}$ induces a (4-)clique in G_A.

<u>Proof</u>. Let $s, t \in S$, with $a^s \bullet a^t = 0$, and let $s = (i_s,j_s,k_s)$, $t = (i_t,j_t,k_t)$. Then $t_1 = (i_s,j_t,k_t)$, $t_2 = (i_t,j_s,k_t)$ and $t_3 = (i_t,j_t,k_s)$ are the only 3 triplets in $S\setminus\{s,t\}$ that satisfy the requirements of the Proposition , i.e., they exist and are unique. Further, $a^s \bullet a^{t_i} = 1$ for $i=1,2,3$ and $a^{t_i} \bullet a^{t_j} = 2$ for all $i,j \in \{1,2,3\}$; hence $\{s,t_1,t_2,t_3\}$ induces a complete subgraph in G_A. To see that this subgraph is maximal, note that any triplet $u \in S\setminus\{s\}$ that contains an element of s, either contains two elements of t (and hence is identical to one of the triplets t_1, t_2 or t_3), or else contains at most one element of t. But then $a^u \bullet a^{t_i} = 0$, where $t_i \in \{t_1,t_2,t_3\}$ is the triplet containing those two elements of t not contained in u (besides the element of s). Thus $\{s,t_1,t_2,t_3\}$ induces a maximal complete subgraph, hence a 4-clique in G_A.||

The set of cliques described in Propositions 2.3 will be called *class 3*. In terms of $K_{n,n,n}$, every class 3 clique of G_A is associated with an ordered pair (s,t) of disjoint triangles of $K_{n,n,n}$, and its node set contains (a) the node of G_A corresponding to the triangle s, and (b) the 3 nodes of G_A corresponding to those triangles t_1, t_2, t_3 of $K_{n,n,n}$ that share 1 node with s and 2 nodes with t.

As to the cardinality of class 3, every ordered pair (s,t) such that $a^s \bullet a^t = 0$ gives rise to a clique of class 3. Since $|S| = n^3$ and for every $s \in S$ that are $(n-1)^3$ indices $t \in S$ such that $a^s \bullet a^t = 0$, the number of ordered pairs (s,t) with $a^s \bullet a^t = 0$ is $n^3(n-1)^3$.

To determine the number of cliques of class 3 we also need to know how many different ordered pairs give rise to the same clique. Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$, and denote by $C(s,t)$ the node set of the clique (of class 3) corresponding to the ordered pair (s,t), i.e. let $C(s,t) := \{s, t_1, t_2, t_3\}$. Further, let $\bar{t}_1 = (i_t, j_s, k_s)$, $\bar{t}_2 = (i_s, j_t, k_s)$, $\bar{t}_3 = (i_s, j_s, k_t)$. Then we have

<u>Proposition 2.4</u>. $C(s,t) = C(t_i, \bar{t}_i)$ for $i = 1,2,3$.

<u>Proof</u>. Consider the ordered pair $(t_1, \bar{t}_1)$. From the definitions, the 4 triplets of the set $C(t_1, \bar{t}_1)$ are $(i_s, j_t, k_t) = t_1$, $(i_s, j_s, k_s) = s$, $(i_t, j_t, k_s) = t_3$, and $(i_t, j_s, k_t) = t_2$; thus $C(t_1, \bar{t}_1) = C(s,t)$. By symmetry, $C(t_i, \bar{t}_i) = C(s,t)$ for $i = 2,3$.||

<u>Corollary 2.5</u>. The number of cliques of class 3 is $n^3(n-1)^3/4$.

<u>Proof</u>. Every clique of class 3 arises from 4 distinct ordered pairs, and the number of the latter is $n^3(n-1)^3$.||

<u>Proposition 2.6</u>. G_A is regular of degree $3n(n-1)$ and has $\frac{3}{2} n^4(n-1)$ edges.

<u>Proof</u>. Let a^s be an arbitrary column of A. There are $(n-1)^3$ columns a^t of A such that $a^s \cdot a^t = 0$, hence there are $n^3-1-(n-1)^3 = 3n(n-1)$ columns a^u of A such that $a^s \cdot a^u \neq 0$. Thus the degree of node s in G_A is $3n(n-1)$, and by symmetry this is true of all $s \in S$. Since the number of edges of a graph is one half of the sum of the degrees of its nodes, G_A has $\frac{1}{2} \times n^3 \times 3n(n-1) = \frac{3}{2} n^4(n-1)$ edges.||

Next we show that G_A has no other cliques than the ones described above.

<u>Theorem 2.7</u>. The only cliques of G_A are those of classes 1, 2 and 3.

<u>Proof</u>. Let C be any clique of G_A and let $t = (i_o, j_o, k_o) \in C$. If each $w \in C$ meets t in at least two indices then $C = C(t)$, i.e. C belongs to class 2. Otherwise there is an $s \in C$ that meets t in only one index. Suppose

w.l.o.g. $s = (i_0, j_1, k_1)$, $j_1 \neq j_0$, $k_1 \neq k_0$. If every $w \in C$ contains i_0, then $C = S^{i_0}$ i.e. C belongs to class 1; otherwise there is a $w \in C$ that meets t in an index other than i_0. W.l.o.g., suppose $w = (i_1, j_o, k_1)$, $i_1 \neq i_o$. If $r = (i_1, j_1, k_o) \in C$ then $C = C(t,(i_1,j_1,k_1))$, i.e. C belongs to class 3. If $r \notin C$ then every element of C must contain two of the indices of $q = (i_0, j_0, k_1)$, and $C = C(q)$, i.e. again C belongs to class 2.||

3. Facets of P_I Induced by Cliques of G_A.

If C is the vertex set of a clique of G_A, then obviously every $x \in P$ satisfies the inequality

(3.1) $\sum(x_s: s \in C) \leq 1$

Such inequalities are known to define facets of $\tilde{P}_I$, the set packing polytope associated with P_I[17]; but Since P_I itself is a face of $\tilde{P}_I$, it is an open question whether an inequality (3.1) also defines a facet of P_I. In this section we answer this question exhaustively.

First, some definitions and basic concepts. For any polyhedron *P*, let dim *P* denote the dimension of *P* (defined as the dimension of the affine hull of *P*, i.e. of the smallest subspace containing *P*). An inequality $\pi x \leq \pi_o$ is said to define a *facet* of *P*, if it is satisfied by every $x \in P$ and the polyhedron $P^{\pi} := \{x \in P | \pi x = \pi_o\}$ has dimension dim P-1. If $\pi x = \pi_o$ for all $x \in P$, the inequality $\pi x \leq \pi_o$ is said to define an *improper face* of *P*. In this case of course dim P^{π} = dim *P*. To show that $\pi x \leq \pi_o$ does not define an improper face, it is sufficient to exhibit a point $x \in P$ such that $\pi x < \pi_o$. Once this is ascertained to be the case, dim $P^{\pi} \leq$ dim P-1, since (a) dim *P* is the number of variables in the system defining *P*, minus the rank of the equality system of *P* (i.e. of the system of linear *equations* satisfied by all $x \in P$); and (b) the addition of the equation $\pi x = \pi_o$, not implied

by the system defining P, increases the rank of the equality system by at least 1. Thus showing that $\pi x \leq \pi_o$ defines a facet of P essentially amounts to showing that the dimension of P^{π}, known to be bounded by dim P-1, is actually equal to this bound. The most commonly used procedure for doing this is to exhibit dim P affinely independent points $x \in P^{\pi}$. Another approach is to show that the addition of $\pi x = \pi_o$ to the constraints defining P increases the rank of the equality system of P by exactly one; in other words, that *any* equation satisfied by all $x \in P^{\pi}$ is a linear combination of the equations in the system defining P^{π}. In this paper we will take the latter approach, and will use it also to establish the dimension of P_I itself. We will implement this approach via a technique similar to that used by Maurras [16], as well as by Cornuejols and Pulleyblank [5], (see also Cornuejols and Thizy [6]).

We first establish the dimension of P_I.

Let P denote the feasible set of the linear programming relaxation of P_I, i.e.

$$P = \{x \in R^{n^3} | Ax = e, x \geq 0\}.$$

Lemma 3.1. The rank of the system $Ax = e$ is 3n-2.

Proof. The rank of $Ax = e$ is at most 3n-2, since equation 2n is the sum of the first n equations, minus the sum of equations n+1,...,2n-1; and equation 3n is the sum of the first n equations minus the sum of equations 2n+1,...,3n-2. On the other hand, the rank of $Ax = e$ is at least 3n-2, since we can exhibit 3n-2 affinely independent columns of A. Consider the three sets of columns indexed by the following triplets:

(2,1,1), (3,1,1), ..., (n,1,1);

(1,2,1), (1,3,1), ..., (1,n,1);

(1,1,1), (1,1,2), ..., (1,1,n)

The first two sets contain n-1 columns each, the last one contains n columns. The matrix formed by these columns (in the order of their listing), after deletion of the first row of set I and the first row of set J, becomes a square lower triangular (hence nonsingular) matrix of order 3n-2, with each diagonal element equal to 1.||

A direct consequence of Lemma 3.1 is the following known result.

Proposition 3.2. $\dim P = n^3 - 3n + 2$.

Proof. The dimension of P is the number of variables in its defining system (n^3), minus the rank of its equality system $Ax = e$ (3n-2). ||

We are interested in $\dim P_I$. Since $P_I \subset P$, $\dim P_I \leq n^3-3n+2$, and strict inequality holds if and only if there exists an equation $\alpha x = \alpha_0$ satisfied by all $x \in P_I$, that is not implied by (not a linear combination of) the equations $Ax = e$. We will show that no such equation exists.

Theorem 3.3. Let $n \geq 3$, and suppose every $x \in P_I$ satisfies $\alpha x = \alpha_0$ for some $\alpha \in R^{n^3}$, $\alpha_0 \in R$. Then there exist scalars λ_i, $\forall\, i \in I$, μ_j, $\forall\, j \in J$, and ν_k, $\forall\, k \in K$, satisfying

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k,\ \forall (i,j,k) \in I \times J \times K,$$

(3.2)

$$\alpha_0 = \sum(\lambda_i : i\epsilon I) + \sum(\mu_j : j\epsilon J) + \sum(\nu_k : k\epsilon K)$$

Proof. Define $\lambda_i = \alpha_{i11} - \alpha_{111}$, $\mu_j = \alpha_{1j1} - \alpha_{111}$, $\nu_k = \alpha_{11k}$.

We will show that

$$\begin{aligned}\alpha_{ijk} &= \lambda_i + \mu_j + \nu_k \\ &= \alpha_{i11} + \alpha_{1j1} + \alpha_{11k} - 2\alpha_{111}.\end{aligned}$$

This is clearly true for α_{111}, α_{i11}, α_{1j1} and α_{11k}. For α_{1jk}, $j \neq 1$, $k \neq 1$ we claim that $\alpha_{1jk} = \alpha_{1j1} + \alpha_{11k} - \alpha_{111}$. Consider $x \in P_I$ such that $x_{111} = x_{ij\ell} = 1$, $i \neq 1 \neq \ell$. Define x' by $x'_{111} = x'_{ij\ell} = 0$, $x'_{1j1} = x'_{i1\ell} = 1$ and $x'_t = x_t$ otherwise. We will call the

construction of x' from x a second index interchange on the triplets (1,1,1) and (i,j,ℓ) (first and third index interchanges are defined analogously).

Let $\bar{x} \in P_I$ be such that $\bar{x}_{1jk} = \bar{x}_{i1\ell} = 1$, and construct $\bar{x}'$ from $\bar{x}$ by a second index interchange on $(1,j,k)$ and $(i,1,\ell)$. Since $\alpha x = \alpha x'$ and $\alpha\bar{x} = \alpha\bar{x}'$ we have

$$\alpha_{111} + \alpha_{ij\ell} = \alpha_{1j1} + \alpha_{i1\ell} \quad \text{and} \quad \alpha_{1jk} + \alpha_{i1\ell} = \alpha_{11k} + \alpha_{ij\ell}.$$

Adding these two equations and canceling terms gives $\alpha_{111} + \alpha_{1jk} = \alpha_{1j1} + \alpha_{11k}$ or

$$\alpha_{1jk} = \alpha_{1j1} + \alpha_{11k} - \alpha_{111} \tag{3.3}$$

as required. The cases α_{i1k} and α_{ij1} follow by symmetry.

For α_{ijk}, $i \neq 1$, $j \neq 1$, $k \neq 1$, consider $x \in P_I$ with $x_{111} = x_{ijk} = 1$ and define x' from x by a first index interchange on (1,1,1) and (i,j,k). Then, as above, $\alpha_{111} + \alpha_{ijk} = \alpha_{i11} + \alpha_{1jk}$. Substituting for α_{1jk} its value given by (3.3) yields $\alpha_{111} + \alpha_{ijk} = \alpha_{i11} + \alpha_{1j1} + \alpha_{11k} - \alpha_{111}$, or

$$\alpha_{ijk} = \alpha_{i11} + \alpha_{1j1} + \alpha_{11k} - 2\alpha_{111} \tag{3.4}$$

as required.

Finally, let $\hat{x}$ be defined by $\hat{x}_{ijk} = 1$ if $i = j = k$, $\hat{x}_{ijk} = 0$ otherwise. Then $\hat{x} \in P_I$, hence $\alpha\hat{x} = \alpha_0$, or $\alpha_0 = \sum(\lambda_i : i\epsilon I) + \sum(\mu_j : j\epsilon J) + \sum(\nu_k : k\epsilon K)$. ||

<u>Corollary 3.4</u>. For $n \geq 3$, $\dim P_I = n^3 - 3n + 2$.

<u>Proof</u>. From Theorem 3.3, if $n \geq 3$ then the smallest affine subspace containing P_I is the one defined by the system $Ax = e$; the dimension of P_I is therefore the same as that of P. ||

Next we turn to the constraints defining P and ask the question, which ones among these define facets of P_I.

<u>Theorem 3.5</u>. Every inequality $x_s \geq 0$ for some $s \in S$ defines a facet of P_I.

<u>Proof</u>. The statement is true if and only if the polytope $P_I^s = \{x \in P_I | x_s = 0\}$ has dimension $\dim P_I - 1 = n^3 - 3n + 1$. Clearly, $\dim P_I^s \leq n^3 - 1 - r$, where r is the rank of the system $A^s x = e$, and A^s is the matrix obtained from A by removing the column a^s. The rank of A^s is easily seen to be the same as the rank of A, i.e. $r = 3n-2$. This is immediate in the case when a^s is not among those columns used in the proof of Lemma 3.1, and follows by symmetry for the other case. Hence the dimension of P_I^s is at most $n^3 - 3n + 1$. To prove that this bound is actually attained, one can use the same argument as in the proof of Theorem 3.3 to show that any equation $\alpha x = \alpha_o$ (other than $x_s = 0$) satisfied by every $x \in P_I^s$ is a linear combination of the equations $A^s x = e$. The argument goes through essentially unchanged.||

The inequalities $x_s \leq 1$ of course do not define facets, since they are implied by $Ax = e$. In fact, it is not hard to see that each inequality $x_s \leq 1$ defines a (n^3-3n^2+4)-dimensional face of P_I. Indeed, if P_I^k denotes the polyhedron P_I for $n = k$, then $P_I^n \cap \{x | x_s = 1\} = P_I^{n-1}$, and from Corollary 3.2, $\dim P_I^{n-1} = n^3-3n^2+4$ for all $n \leq 3$.

We now turn to the inequalities (3.1) defined by the cliques of G_A.

Each clique of class 1 induces an inequality whose left hand side coefficient vector is one of the rows of A. Hence each such inequality is satisfied with equality by every $x \in P_I$ and therefore defines an improper face of P_I.

Next we consider the inequalities (3.1) induced by the cliques of class 2. Each clique in this class is defined relative to some index (triplet) $s \in S$, and has a node set of the form $C(s) := \{s\} \cup T(s)$ (see Proposition 2.2). It is not hard to see, that the inequality (3.1) induced by the clique of class 2 defined relative to $s = (i,j,k)$ can be obtained by adding up the

three equations of $Ax = e$ indexed by i,j,k, dividing the resulting equation by 2, then replacing $=$ by $\leq$ and rounding down each coefficient to its nearest integer. In other words, these inequalities belong to the elementary closure of the system $Ax = e$, $x \geq 0$, as defined by Chvátal [4]. The proof of the next theorem will be deferred to section 6, where a more general class of inequalities belonging to the elementary closure of $Ax = e$, $x \geq 0$ and having left hand side coefficients equal to 0 or 1, will be shown to be facet inducing for P_I.

Theorem 3.6. For $n \geq 3$, the inequality

(3.5) $\quad \sum(x_t\colon t \varepsilon C(s)) \leq 1$

defines a facet of P_I for every $s \varepsilon S$.

Finally, we turn to the inequalities (3.1) induced by cliques of class 3. Remember that each clique in this class is defined relative to an ordered pair (s,t) of disjoint triplets, and has a node set of the form $\{s,t_1,t_2,t_3\}$, where each t_i, $i = 1,2,3$, contains one element of s and two elements of t (see Proposition 2.3). Let $C(s,t)$ denote the node set of the clique of class 3 defined relative to the ordered pair (s,t).

Theorem 3.7. For $n \geq 4$, the inequality

(3.6) $\quad \sum(x_u\colon u \varepsilon C(s,t)) \leq 1$

defines a facet of P_I for all $s,t \varepsilon S$.

Proof: W.l.o.g., let $s = (n,n,n)$ and $t = (p,q,r)$, with $p,q,r < n$. The inequality (3.6) does not define an improper face of P_I, since it holds strictly, for instance, for the vector x defined by $x_q = 1$, $q = (p + \alpha,\ q + \alpha,\ r + \alpha) \pmod n$, $\alpha = 1,\ldots,n$, $x_q = 0$ otherwise.

Now let

$$P_I^{C(s,t)} := \operatorname{conv}\{x \,\varepsilon\, \{0,1\}^{n^3} \mid Ax = e,\ \sum_{s \,\varepsilon\, C(s,t)} x_s = 1\}.$$

To show that (3.6) defines a facet of P_I, i.e., that $\dim P_I^{C(s,t)} = \dim P_I - 1$, we use the same approach as for Theorem 3.3, i.e., we exhibit scalars λ_i, $i\varepsilon I$, μ_j, $j\varepsilon J$, ν_k, $k\varepsilon K$ and π such that if $\alpha x = \alpha_0$ for all $x\varepsilon P_I^{C(s,t)}$, then

$$(3.7) \qquad \alpha_{ijk} = \begin{cases} \lambda_i + \mu_j + \nu_k & (i,j,k)\varepsilon S\backslash C(s,t) \\ \lambda_i + \mu_j + \nu_k + \pi & (i,j,k)\varepsilon C(s,t) \end{cases}$$

and

$$(3.8) \qquad \sum(\lambda_i\colon i\varepsilon I) + \sum(\mu_j\colon j\varepsilon J) + \sum(\nu_k\colon k\varepsilon K) + \pi.$$

Again, we define $\lambda_i = \alpha_{i11} - \alpha_{111}$, $i\varepsilon I$, $\mu_j = \alpha_{1j1} - \alpha_{111}$, $j\varepsilon J$, $\nu_k = \alpha_{11k}$, $k\varepsilon K$. Then for α_{111}, α_{i11}, α_{1j1} and α_{11k} the first equation of (3.7) clearly holds. For α_{1jk}, $j \neq 1 \neq k$, we have to show that

$$(3.9) \qquad \alpha_{1jk} = \alpha_{1j1} + \alpha_{11k} - \alpha_{111}.$$

If $j \neq n \neq k$, let $x\varepsilon P_I^{C(s,t)}$ be such that $x_{111} = x_{ij\ell} = x_{nnn} = 1$, $\ell \neq 1$, and let $\bar{x}\varepsilon P_I$ be such that $\bar{x}_{1jk} = \bar{x}_{i1\ell} = \bar{x}_{nnn} = 1$. Then *performing second* index interchanges on $(1,1,1)$, (i,j,ℓ) and on $(1,j,k)$, $(i,1,\ell)$, respectively, produces x', $\bar{x}'\varepsilon P_I^{C(s,t)}$, and from $\alpha x = \alpha x'$ and $\alpha\bar{x} = \alpha\bar{x}'$ we obtain two equations whose sum yields (3.8). The procedure is analogous for the other cases, namely: if $j = n = k$, we use x, $\bar{x}$ such that $x_{111} = x_{ij\ell} = x_{nqr} = 1$, $\bar{x}_{1jk} = \bar{x}_{i1\ell} = \bar{x}_{nqr} = 1$; if $j = n$, $k \neq n$, use x, $\bar{x}$ with $x_{111} = x_{ij\ell} = x_{pqn} = 1$, $\bar{x}_{1jk} = \bar{x}_{i1\ell} = \bar{x}_{pqn} = 1$; and finally, if $j \neq n$, $k = n$, reverse the roles of the second and third index.

Since (3.9) holds for all α_{1jk}, by symmetry an analogous relation holds for all α_{i1k} and α_{ij1}.

Next consider any $(i,j,k)\varepsilon C(s,t)$, and define

$$(3.10) \qquad \pi_{ijk} = \alpha_{ijk} - \lambda_i - \mu_j - \nu_k.$$

To prove (3.7), we have to show that all π_{ijk} are equal. Note that for $(i,j,k)\varepsilon C(s,t)$, we only have to consider the cases π_{njk}, π_{ink}, π_{ijn}.

Let $x \varepsilon P_I^{C(s,t)}$ be such that $x_{nnn} = x_{pqr} = 1$, and define x' from x by a first index interchange on (n,n,n) and (p,q,r). Then $x' \varepsilon P_I^{C(s,t)}$ follows from $x \varepsilon P_I^{C(s,t)}$, and $\alpha x = \alpha x'$ implies $\alpha_{nnn} + \alpha_{pqr} = \alpha_{pnn} + \alpha_{nqr}$. Since (n,n,n), (n,q,r)$\varepsilon$C(s,t) and (p,q,r), (p,n,n)$\not\varepsilon$C(s,t), substituting for α_{nnn}, α_{nqr} and for α_{pqr}, α_{pnn} their values defined by (3.10) and (3.7), respectively, we obtain

$$\pi_{nnn} + \lambda_n + \mu_n + \nu_n + \gamma_p + \mu_q + \nu_r =$$
$$\pi_{nqr} + \lambda_n + \mu_q + \nu_r + \lambda_p + \mu_n + \nu_n$$

or $\pi_{nnn} = \pi_{nqr} =: \pi$. By symmetry, we also have $\pi_{pnr} = \pi_{pqn} = \pi$.

Finally, let x* be defined by $x^*_{ijk} = 1$ if $i = j = k$, $x^*_{ijk} = 0$ otherwise. Then $x^* \varepsilon P_I^{C(s,t)}$, hence $\alpha x^* = \alpha_0$, and we have (3.8).||

Unlike the cliques of class 2, those of class 3 do not belong to the elementary closure of the system $Ax = e$, $x \geq 0$, i.e., in the terminology of [4], they are not of rank 1.

<u>Proposition 3.8</u>. The inequalities (3.6) are of rank 2.

<u>Proof</u>: Every inequality associated with a class 3 clique C(s,t) can be obtained by the following procedure. Let $s = (i_s, j_s, k_s)$ and $t = (i_t, j_t, k_t)$. Add the equations of $Ax = e$ indexed by i_s, j_s, k_s and twice the clique inequality of class 2 associated with t; divide the resulting inequality by 3 and round down all coefficients to the nearest integer. Since the constraints used in this procedure are of rank 0 or rank 1, the resulting inequality is of rank 2.||

<u>Theorem 3.9</u>. The inequalities (3.1) corresponding to distinct cliques define distinct facets (faces in the case of type 3 cliques with n=3) whenever $n \geq 3$.

<u>Proof</u>. For any two cliques C_1 and C_2, there is a feasible solution x with $x_r = x_s = x_t = 1$, such that x has the following properties: 1.

$s \in C_1 \backslash C_2$; 2. $t \in C_2 \backslash C_1$; 3. $r \in S \backslash (C_1 \cup C_2)$; 4. if $C_2 = C(t_o)$ then $t \neq t_0$ and $t_o \cap r = \emptyset$; if $C_2 = C(t, r_0)$ then $r \neq r_0$. This can be shown by direct construction of such a solution for each of several subcases of the following three cases: (1) C_1 and C_2 are both of type 2; (2) C_1 is of type 2 and C_2 of type 3; (3) C_1 and C_2 are both of type 3.

It is easy to see that if x satisfies conditions 1-4 then

$$\sum(x_s\colon s \in C_1) = 1, \quad \sum(x_t\colon t \in C_2) = 0,$$

and that an appropriate interchange on r and t produces x' such that

$$\sum(x'_s\colon s \in C_1) = 0, \quad \sum(x'_t\colon t \in C_2) = 1. ||$$

4. Detecting Violated Clique-Facets

It is of great interest in terms of algorithm development to be able to determine, for an arbitrary noninteger solution to the LP-relaxation of an integer program, whether that solution violates a facet of the convex hull of integer solutions. One may solve the LP-relaxation, then identify a facet-defining inequality that cuts off the solution obtained and either add it to the constraint set of the IP, or take it into the objective function with a Lagrange multiplier. In general, for an NP-hard problem the facet-identification problem is also NP-hard, but for some subsets of the facets it may be possible to efficiently identify which, if any, members of the subsets are violated by an IP solution. Recent efforts to implement algorithms based on this strategy (and employing branch-and-bound techniques when a fractional solution is reached that does not violate any of the facets under consideration) have met with marked success [1], [8]. In this section we describe an efficient algorithm for detecting clique-facets violated by an arbitrary $x \in P$, i.e., an arbitrary solution to the LP-relaxation of AP3.

Although the cardinality of the set of clique-facets is $O(n^6)$, (namely, n^3 facets from cliques of class 2, and $n^3(n-1)^3/4$ from cliques of class 3), the proposed algorithm can be shown to have a worst-case running time of $O(n^4)$. In terms of the number $|S|$ of variables, this is $O(|S|^{4/3})$.

We first remark that given a noninteger $x \in P$, it can be detected in $O(n^4)$ steps whether any inequality induced by a clique of class 2 is violated. Indeed, each of the n^3 cliques of class 2 is associated with some $s \in S$, and is induced by a node set of the form $\{s\} \cup T(s)$, where $T(s)$ is the set of those triplets that differ from the the triplet s in exactly one element. Since the cardinality of $T(s)$ is $3(n-1)$, for each $s \in S$ it requires $O(n)$ steps to identify and add all x_{ijk} such that $(i,j,k) \in C(s)$, in order to check whether the sum exceeds 1 (in which case the corresponding inequality is violated) or not. To execute this for all $s \in S$ therefore requires $O(n \times n^3) = O(n^4)$ steps.

For cliques of class 3 (whose number is $O(n^6)$) the complexity bound is not so straightforward. However, we will give an algorithm which performs that task too in $O(n^4)$ steps. This is possible due to the following fact: Each clique of class 3 is of cardinality 4; therefore any $x \in P$ that violates some inequality induced by a clique of class 3 must have at least one component of value $\geq 1/4$. On the other hand, we have

<u>Lemma 4.1</u> For any $x \in P$ and any positive integer k, the number of components with value $\geq 1/k$ is $\leq kn$.

<u>Proof</u>. The value of the linear program

(L) $\max \{ex \mid x \in P\}$

is easily seen to be n, since the vectors $x \in R^{n^3}$ and $u \in R^{3n}$, defined by $x_s = 1/n^2$, $\forall\, s \in S$ and $u_r = 1/3$, $\forall\, r \in R$, are feasible solutions to (1) and its dual, respectively, with the common value of n; hence they are optimal.

Now if x has more than kn components with value $\geq 1/k$, then $ex > n$, a contradiction.||

<u>Theorem 4.2</u>. It can be determined in $O(n^4)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 3.

<u>Proof</u>. Let $C(s,t)$ be the node set of a clique of class 3. Since $|C(s,t)| = 4$, if $x \in P$ violates the facet-inequality corresponding to $C(s,t)$, then from Lemma 4.1 x has at least one component $\geq 1/4$. Further, if $C(s,t) = \{s,t_1,t_2,t_3\}$, from Proposition 2.4 there is no loss of generality in assuming that this happens for the component indexed by s, i.e., that $x_s \geq 1/4$. Thus, instead of examining all ordered pairs (s,t) such that $a^s \bullet a^t = 0$, we can restrict ourselves to examining those ordered pairs (s,t) such that $x_s \geq 1/4$ and $a^s \bullet a^t = 0$.

Consider now the following algorithm.

1. Order S according to nonincreasing values of x_s, $s \in S$.

2. For each of the first 4n elements $s = (i_s,j_s,k_s)$ of the ordered set S such that $x_s > 1/4$ and each of the $(n-1)^3$ triplets $t = (i_t,j_t,k_t) \in S$ such that $i_t \neq i_s$, $j_t \neq j_s$ and $k_t \neq k_s$, calculate the sum $\Sigma(s,t) = x_{i_s j_s k_s} + x_{i_s j_t k_t} + x_{i_t j_s k_t} + x_{i_t j_t k_s}$. If $\Sigma(s,t) > 1$, stop: the inequality associated with (s,t) is violated; otherwise continue.

Since the algorithm examines all pairs (s,t) such that $a^s \bullet a^t = 0$ and $x_s > 1/4$, it either finds a pair whose corresponding facet inequality is violated by x, or it stops with the conclusion that x satisfies all facet-inequalities induced by cliques of class 3. Step 1 is executed once and it requires $O(n^3 \log n^3)$ operations. Step 2 is executed at most $4n(n-1)^3$ times, and each execution requires 3 additions. Hence, the overall complexity of the algorithm is $O(n^4)$.||

5. The Odd Holes of G_A

In this section we describe the odd holes (odd-length chordless cycles) of G_A and discuss some of their properties.

Proposition 5.1. A node set $H \subseteq S$ such that $|H| = 2p + 1$ for some positive integer $p \geq 2$ induces an odd hole in G_A if and only if H can be ordered into a sequence $\{s_1,\ldots,s_{2p+1}\}$ such that for all s_r, $s_t \in H$,

$$(5.1) \qquad a^{s_r} \bullet a^{s_t} = \begin{cases} 1 \text{ or } 2 & \text{if } t = r \pm 1 \pmod{2p+1} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Since two distinct columns of A have at most two 1's in common, $1 \leq a^{a_r} \bullet a^{s_t} \leq 2$ if and only if s_r and s_t are adjacent.||

For an odd hole $H = \{s_1,\ldots,s_{2p+1}\}$, the *link* of a pair (s_r, s_{r+1}) is the row (or pair of rows) of A that contains the common 1's of a^{s_r} and $a^{s_{r+1}}$. We say that (s_r, s_{r+1}) has a ***single link*** (a ***double link***) if $a^{s_r} \bullet a^{s_{r+1}} = 1$ (=2). Single links are in I, J or K, whereas double links are in $I \cup J$, $I \cup K$ or $J \cup K$: no double link can be in a single ground set.

Proposition 5.2. No two adjacent edges of an odd hole have their links in the same ground set.

Proof; If (s_r, s_{r+1}) and (s_{r+1}, s_{r+2}) have links in the same ground set, then $a^{s_r} \bullet a^{s_{r+2}} \geq 1$, contrary to (5.1).||

Since there are only three ground sets, it follows that no two adjacent edges can have double links.

Proposition 5.3. The number d of double links of an odd hole of length $2p + 1$ satisfies

$$(5.2) \qquad \max\{0, 2(2p+1) - 3n\} \leq d \leq p - 1$$

Proof: The number of positive components of the vector $a^{\Sigma} := a^{s_1} + \ldots + a^{s_{2p+1}}$ is $2(2p+1) - d$, and this number cannot exceed that of the rows of A,

i.e., $2(2p+1) - d \le 3n$. Also, $d \ge 0$. This proves the validity of the lower bound.

To see that $d \le p - 1$, suppose H is a (2p+1)-hole that has p double links, say for $(s_1, s_2), (s_3, s_4), \ldots, (s_{2p-1}, s_{2p})$. Assume w.l.o.g. that (s_1, s_2) has its (double) link in $I \cup J$; then (s_2, s_3) has its link in K, and therefore (s_3, s_4) again has its link in $I \cup J$. By the same reasoning, each double link in the above sequence is in $I \cup J$. But then the (single) links of (s_{2p}, s_{2p+1}) and (s_{2p+1}, s_1) both have to be in K, a contradiction.||

<u>Proposition 5.4</u>. The maximum length of an odd hole in G_{A_n} is $2n - 1$. All odd holes of maximum length have $n - 2$ double links. For $p < n - 1$, G_{A_n} has odd holes of length $2p + 1$ with d double links for every integer d satisfying (5.2).

<u>Proof</u>: From Proposition 5.3, $2(2p+1) \le 3n + d$, where $d \le p - 1$. The maximum of p is thus $n - 1$, and the maximum of $2p + 1$ is $2n - 1$. Since this maximum is attained for $d = p - 1$, which for $p = n - 1$ is both an upper and a lower bound on d, all odd holes of maximum length have $p - 1 = n - 2$ (i.e., the maximum number of) double links.

For $p < n - 1$, if H is any (2p+1)-hole with $d > \max\{0, 2(2p+1) - 3n\}$, one can obtain from H a (2p+1)-hole H' with $d' = d - 1$ double links, by taking any doubly linked pair (s_r, s_{r+1}), and replacing s_r in H with some $s_* \in S\backslash H$ such that $a^{s_*} \bullet a^{s_{r-1}} = a^{s_*} \bullet a^{s_{r+1}} = 1$ and $a^{s_*} \bullet a^{s_i} = 0$ for all $i \in \{1, \ldots, 2p + 1\} \setminus \{r - 1, r, r + 1\}$. Two of the three 1's of a^{s_*} are given by the above condition, and the third one can be in any row i such that $a_i^s = 0$ for all $s \in H$. Thus s_* exists whenever the number $2(2p+1) - d$ of positive components of a^{Σ} is less than 3n. Since by assumption $d > 2(2p+1) - 3n$, s_* with the required properties exists.||

Thus one can distinguish between different types of odd holes of a given length, according to the number of their double links. For n = 3, the only odd holes of G_A are those of length 5(=2n-1) and they all have a maximum number (n-2=1) of double links. For n = 4, G_A has odd holes of lengths 5 and 7. The 5-holes can have one double link or none; the 7-holes all have two (=n-2) double links. For n = 5, G_A had odd holes of length 5, 7 and 9. While the 9-holes all have 3 double links, the 7-holes can have 2, 1 or 0 double links and the 5-holes can have 1 or 0 double links.

Fig. 5.1 shows some of the 7-holes of G_{A_5}. The numbered circles are the nodes $s \in H$; the lines represent the (single or double) links; the symbol on each line stands for the associated row of A.

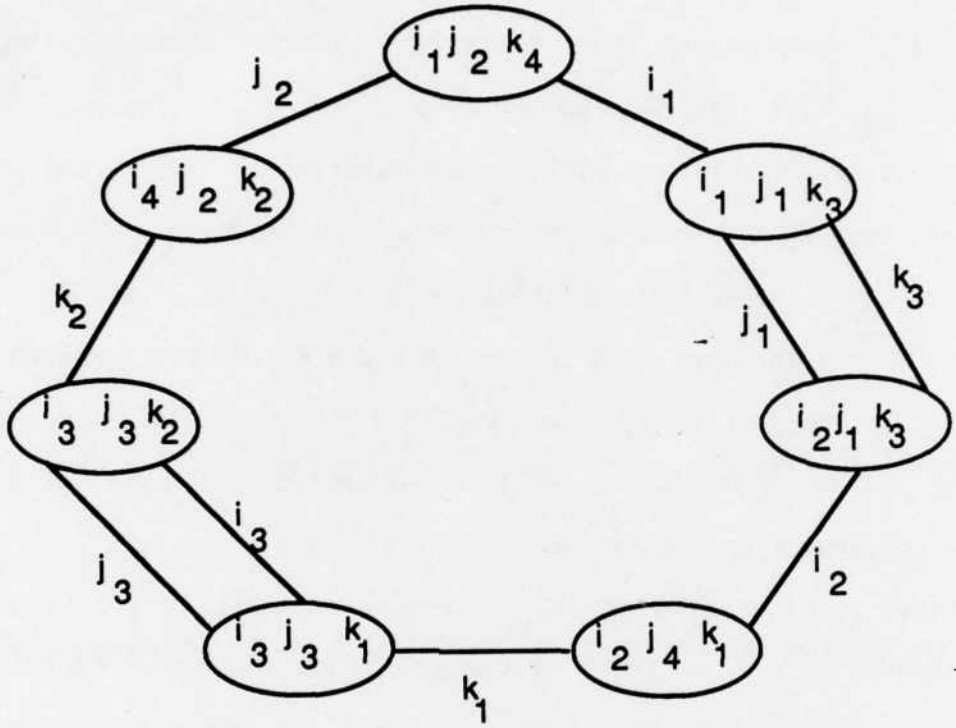
Next we describe the connection between odd holes of G_A and certain row sets of A. Recall that R and S denote the row and column sets, respectively, of A. for any $Q \subset R$ and $T \subset S$, let A_Q^T denote the submatrix of A with rows and columns indexed by Q and T, respectively. Also, let $A_Q := A_Q^S$ and $A^T := A_R^T$. For any $Q \subseteq R$ and for L = I, J, K, let $Q_L := Q \cap L$. Finally, let C_k^2 denote the circulant matrix of order k with exactly two 1's in each row and column, and 0's everywhere else.

<u>Proposition 5.4</u>. Let H be the node set of an odd hole in G_A, $|H| = 2p + 1$, with d double links. Then A^H has 2^d distinct row sets Q, $|Q| = 2p + 1$, such that

(i) $A_Q^H = C_{2p+1}^2$ up to row and column permutations;

(ii) $1 \le |Q_L| \le p$, L = I, J, K;

(iii) the rows of $A_{R\setminus Q}^H$ are either distinct copies of rows of C_{2p+1}^2 corresponding to double links, or else contain at most one 1.



$i_1 j_2 k_4$
j_2
i_1
$i_4 j_2 k_2$
$i_1 j_1 k_3$
k_2
k_3
j_1
$i_3 j_3 k_2$
$i_2 j_1 k_3$
i_3
j_3
i_2
$i_3 j_3 k_1$
$i_2 j_4 k_1$
k_1

Fig. 5.1

<u>Proof</u>: Let $H = \{s_1,\ldots,s_{2p+1}\}$. The rows of A^H containing the common 1's of a^{s_1} and a^{s_2}, of a^{s_2} and $a^{s_3},\ldots,$ and of $a^{s_{2p+1}}$ and a^{s_1}, form a set of cardinality $2p + 1 + d$, where d is the number of double links. This set contains 2^d subsets Q of cardinality $2p + 1$, obtained by choosing one member of each pair of rows corresponding to a double link of H, plus each row corresponding to a single link. Each such subset forms a square submatrix A_Q^H of order $2p + 1$ that has exactly two ones in every row and column, hence becomes C_{2p+1}^2 after row and column permutations. This proves (i).

If $Q_L = \phi$ for, say, $L = I$, then $Q \subseteq J \cup K$ and every column of A_Q^H has a 1 in Q_J and a 1 in Q_K, contrary to the stated equivalence of A_Q^H and C_{2p+1}^2. Thus $|Q_L| \geq 1$. If, on the other hand, $|Q_L| \geq p + 1$, then $A_{Q_L}^H$ has $2(p+1)$ columns, a contradiction. Thus (ii) holds.

Finally, if any row of $A_{R\setminus Q}^H$ has two or more 1's and is not a copy of some row of A_Q^H, then (5.1) is violated. Further, only those rows of A_Q^H can have copies, whose 1's occur in columns corresponding to a doubly linked pair; in which case the copy is unique.||

While the last Proposition deals with the row sets of A that can be associated with a given odd hole, our next statement concerns the collection of odd holes that can be associated with a given row set.

<u>Theorem 5.5</u>. Let $Q \subset R$, $|Q| = 2p + 1$ for some integer p satisfying $2 \leq p \leq n - 1$, and let $1 \leq |Q_L| \leq p$ for $L = I, J, K$. Then there exists a $(2p+1)$-hole whose set of links contains Q and that has d double links, for every d satisfying (5.2).

<u>Proof</u>: Let $Q_I = \{i_1,\ldots,i_r\}$, $Q_J := \{j_1,\ldots,j_s\}$, $Q_K = \{k_1,\ldots,k_t\}$, and w.l.o.g. let $p \geq r \geq s \geq t \geq 1$. We first identify a family of circulants C_{2p+1}^2 contained in the row set Q, then show how to find the corresponding odd holes of G_A with the desired number of double links.

Consider a sequence of links of the form

$$\underbrace{i\ j\ i\ j\ .\ .\ .\ i\ j\ i}_{x}\ \underbrace{k\ i\ k\ i\ .\ .\ .\ k\ i\ k}_{y}\ \underbrace{j\ k\ j\ k\ .\ .\ .\ j\ k\ j}_{z}$$

where x, y and z denote the numbers of elements in each of the three subsequences formed of elements of $Q_I \cup Q_J$, $Q_K \cup Q_I$ and $Q_J \cup Q_K$, respectively.

The numbers x, y and z satisfy

$$\begin{aligned} x + y + z &= 2p + 1 \\ x + y &= 2r \\ x + z &= 2s \\ y + z &= 2t \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= 2p + 1 - 2t \\ y &= 2p + 1 - 2s \\ z &= 2p + 1 - 2r. \end{aligned}$$

Thus the number of I-links and *J-links in the first subsequence* is $(x-1)/2 + 1 = p - t + 1$ and $(x-1)/2 = p - t$, respectively; the number of K-links and I-links in the second subsequence is $p - s + 1$ and $p - s$, and the number of J-links and K-links in the third subsequence is $p - r + 1$ and $p - r$, respectively. If we take the elements of Q_I, Q_J and Q_K in order, starting with i_1, j_1 and k_1 (which yields one particular family of odd holes), the resulting sequence is

$$\begin{aligned} &(i_1, j_1, \ldots, i_{p-t}, j_{p-t}, i_{p-t+1}, \\ &k_1, i_{p-t+2}, \ldots, k_{p-s}, i_r (= i_{2p+1-t-s}), k_{p-s+1}, \\ &j_{p-t+1}, k_{p-s+2}, \ldots, j_{s-1}, k_t (= k_{2p+1-s-r}), j_s (= j_{2p+1-t-r})). \end{aligned}$$

This sequence specifies two of the three nonzero entries of each column of a (2p+1)-hole. Every choice of the third entry for each column that creates the desired number d of double links gives rise to a (2p+1)-hole with d double links. For instance, if $d = p - 1$, i.e., if we wish to identify the

(2p+1)-hole with the maximum number of double links that has its circulant in Q, we proceed as follows. For every J-link that is between two I-links, we insert a K-link as a second link (this is the only possibility). For this we may use the elements of K immediately following k_t, taken in order. Similarly, for every I-link that is between two K-links, we insert a J-link as a second link, using the elements of J following j_s. Finally, for every K-link that is between two J-links, we insert as a second link an I-link, using the elements of I following i_r. This produces a set of links that determines the third index of all but three columns; and for those three we choose the columns i_{p+1}, j_{p+1}, k_{p+1}. In this fashion we get the (2p+1)-hole H with a maximum number of double links (p-1), whose nonzero coefficients are contained in 3(p+1) rows of A:

$$H = \left\{ \begin{pmatrix} i_1 \\ j_s \\ k_{p+1} \end{pmatrix} \begin{pmatrix} i_1 \\ j_1 \\ k_{t+1} \end{pmatrix} \begin{pmatrix} i_2 \\ j_1 \\ k_{t+1} \end{pmatrix} \cdots \begin{pmatrix} i_{p-t} \\ j_{p-t-1} \\ k_{p-1} \end{pmatrix} \begin{pmatrix} i_{p-t} \\ j_{p-t} \\ k_p \end{pmatrix} \begin{pmatrix} i_{p-t+1} \\ j_{p-t} \\ k_p \end{pmatrix} \begin{pmatrix} i_{p-t+1} \\ j_{p+1} \\ k_1 \end{pmatrix} \begin{pmatrix} i_{p-t+2} \\ j_{s+1} \\ k_1 \end{pmatrix} \cdots \right.$$

$$\left. \cdots \begin{pmatrix} i_{r-1} \\ j_{p-1} \\ k_{p-s} \end{pmatrix} \begin{pmatrix} i_r \\ j_p \\ k_{p-s} \end{pmatrix} \begin{pmatrix} i_r \\ j_p \\ k_{p-s+1} \end{pmatrix} \begin{pmatrix} i_{p+t} \\ j_{p-t+1} \\ k_{p-s+1} \end{pmatrix} \begin{pmatrix} i_{r+1} \\ j_{p-t+1} \\ k_{p-s+2} \end{pmatrix} \cdots \begin{pmatrix} i_{p-1} \\ j_{s-1} \\ k_{t-1} \end{pmatrix} \begin{pmatrix} i_p \\ j_{s-1} \\ k_t \end{pmatrix} \begin{pmatrix} i_p \\ j_s \\ k_t \end{pmatrix} \right\}$$

The construction of the remaining (2p+1)-holes, containing less than the maximum number of double links, is done analogously, except that the third entries of those columns not having a double link can be chosen arbitrarily.||

6. Facets of $\tilde{P}_I$ Associated with the Odd Holes of G_A

It is well known [17] that every odd hole H of G_A gives rise to a facet of the packing polytope associated with H, and that these facets can be lifted into facets of the packing polytope $\tilde{P}_I$ associated with G_A itself. Moreover, the coefficients of the lifted facet inducing inequality depend on the sequence in which the lifting is performed. However, for a general packing polyhedron it is an open question which among its facet inducing odd hole inequalities are also facet inducing for the associated partitioning polyhedron. Also, the lifting procedure is not polynomially bounded.

In this section we describe two classes of lifted odd hole inequalities that are facet inducing for P_I. The first class has all left hand side coefficients equal to 0 or 1, and belongs to the elementary closure of the system $Ax = e$, $x \geq 0$. The second class has left hand side coefficients equal to 0, 1 or 2. Inequalities in both classes can be obtained in time linear in the length of the hole and the number of variables.

<u>Theorem 6.1</u>. Assume $n \geq 3$. Let $Q \subset R$, $|Q| = 2p + 1$ for some integer p satisfying $1 \leq p \leq n - 1$, with $1 \leq |Q_L| \leq p$, $L = I, J, K$, and let

$$(6.1) \qquad S(Q) := \{s \in S \mid \textstyle\sum(a_q^s : q \in Q) \geq 2\}$$

Then the inequality

$$(6.2) \qquad \textstyle\sum(x_s : s \in S(Q)) \leq p$$

defines a facet of P_I.

<u>Proof</u>: As before, let $Q_I = \{i_1,\ldots,i_r\}$, $Q_J = \{j_1,\ldots,j_s\}$, $Q_K = \{k_1,\ldots,k_t\}$, and assume w.l.o.g. that $p \geq r \geq s \geq t \geq 1$. First, (6.2) can be obtained by adding up the equations of $Ax = e$ indexed by Q, dividing the resulting equation by 2, then replacing = with $\leq$ and rounding down the coefficients on both sides of the inequality to the nearest integer. Thus (6.2) is in the elementary closure [4] of $Ax = e$, $x \geq 0$, hence valid.

Next, it is an easy exercise to show that (6.2) does not induce an improper face of P_I, by exhibiting a point $x \in P_I$ which satisfies (6.2) with strict inequality.

Now let

$$P_I^{S(Q)} := \{x \in P_I \mid \sum(x_s: s \in S(Q) = p\}$$

To prove that $P_I^{S(Q)}$ is a facet of P_I, we use the same reasoning as for Theorem 3.3; i.e., we show that any equation $\alpha x = \alpha_0$ satisfied by all $x \in P_I^{S(Q)}$ is a linear combination of the equations $Ax = e$ and $\sum(x_s: s \in S(q)) = 1$

Define $\lambda_i = \alpha_{inn} - \alpha_{nnn}$, $\mu_j = \alpha_{njn} - \alpha_{nnn}$, $\nu_k = \alpha_{nnk}$.

We need to show that there exists a scalar π such that

$$(6.3) \qquad \alpha_{ijk} = \begin{cases} \lambda_i + \mu_j + \nu_k & \text{if } (i,j,k) \notin S(Q) \\ \lambda_i + \mu_i + \nu_\kappa + \pi & \text{if } (i,j,k) \in S(Q) \end{cases}$$

and

$$(6.4) \qquad \alpha_0 = \sum(\lambda_i: i \in I) + \sum(\mu_j: j \in J) + \sum(\nu_k: k \in K) + p\pi.$$

Equation (6.3) clearly holds for α_{nnn}, α_{inn}, α_{njn} and α_{nnk}. For α_{njk}, $j \neq n \neq k$, if $(n,j,k) \notin S(Q)$ then either $j \notin Q$ or $k \notin Q$ or both. Consider x, $\bar{x} \in P_I^{S(Q)}$ such that $x_{nnn} = x_{2j1} = x_{11k} = 1$, $\bar{x}_{njk} = \bar{x}_{2n1} = \bar{x}_{11n} = 1$. (Since $n \geq 3$, such a pair exists). Construct x' from x and $\bar{x}'$ from $\bar{x}$ by a second index interchange (as defined in the proof of Theorem 3.3) on (n,n,n) and $(2,j,1)$, and on (n,j,k) and $(2,n,1)$ respectively. Then x', $\bar{x} \in P_I^{S(Q)}$, and since $\alpha x = \alpha x'$ and $\alpha\bar{x} = \alpha\bar{x}'$, $\alpha_{nnn} + \alpha_{2j1} = \alpha_{njn} + \alpha_{2n1}$ and $\alpha_{njk} + \alpha_{2n1} = \alpha_{nnk} + \alpha_{2j1}$. Adding the last two equations and collecting terms yields

$$(6.5) \qquad \alpha_{njk} = \alpha_{njn} - \alpha_{nnn} + \alpha_{nnk} = \mu_j + \nu_k,$$

which is (6.3) for this case (since $\lambda_n = 0$ by definition). The case of α_{ink} and α_{ijn} is analogous to that of α_{njk}.

Finally, for α_{ijk} with $i \neq n$, $j \neq n$, $k \neq n$, if $(i,j,k) \notin S(Q)$, then at most one of i, j and k belongs to Q. Consider $x \in P_I^{S(Q)}$ such that $x_{nnn} = x_{ijk} = 1$. Since $(n,n,n) \notin S(Q)$ and $(i,j,k) \notin S(Q)$, x needs to have p additional components equal to 1. They can be identified by choosing p nonadjacent elements of a (2p+1)-hole whose set of links includes Q, in such a way as to leave uncovered the row corresponding to $\{i,j,k\} \cap Q$. Then defining x' from x by a first index interchange on (n,n,n) and (i,j,k), we obtain $\alpha_{nnn} + \alpha_{ijk} = \alpha_{inn} + \alpha_{njk}$; and substituting for α_{njk} its value given by (6.5), we have $\alpha_{ijk} = \alpha_{inn} - \alpha_{nnn} + \alpha_{njn} - \alpha_{nnn} + \alpha_{nnk} = \lambda_i + \mu_j + \nu_k$.

This completes the proof of (6.3) for $(i,j,k) \notin S(Q)$.

For $(i,j,k) \in S(Q)$ define

(6.6) $\qquad \pi_{ijk} = \alpha_{ijk} - \lambda_i - \mu_i - \nu_k.$

To show that all π_{ijk} are equal, consider $x \in P_I^{S(Q)}$ such that $x_{rst} = x_{uvw} = 1$, where r, s, t $\in$ Q and u, v, w $\notin$ Q. *Such x clearly exists.* Define x' from x by a first index interchange on (r,s,t) and (u,v,w); then $\alpha_{rst} + \alpha_{uvw} = \alpha_{ust} + \alpha_{rvw}$.

Substituting for α_{rst} and α_{ust} their values given by (6.6) and for α_{uvw} and α_{rvw} their values given by (6.3) we obtain

$$\pi_{rst} + \lambda_r + \mu_s + \nu_t + \lambda_u + \mu_v + \nu_w =$$
$$\pi_{ust} + \lambda_u + \mu_s + \nu_t + \lambda_r + \mu_v + \nu_w$$

or $\pi_{rst} = \pi_{ust}$. By symmetry, $\pi_{rst} = \pi_{rvt} = \pi_{rsw}$ for all r, s, t $\in$ Q and u, v, w $\notin$ Q.

The above reasoning can now be repeated with (r,s,t) and (u,v,w) replaced by (u,s,t) and (r,v,w), and the first index interchange replaced by a second index interchange on (u,s,t) and (r,v,w). This yields $\pi_{ust} = \pi_{uvt}$ and by symmetry $\pi_{ust} = \pi_{usw}$, $\forall$ s, t $\in$ Q and u,v,w $\notin$ Q. It then follows that $\pi_{ijk} = \pi$, $\forall$ $(i,j,k) \in S(Q)$, which completes the proof of (6.3).

Finally, since any $x \in P_I^{S(Q)}$ has exactly p positive components in S(Q) and exactly one positive component for every $i \in I$, $j \in J$ and $k \in K$, substituting the values of α_{ijk} given by (6.2) into the equation $\alpha x = \alpha_0$ for any $x \in P_I^{S(Q)}$ yields (6.4).||

Notice that in Theorem 6.1 we did not require that $p \geq 2$; in other words, (6.2) is a facet inducing inequality also when $p = 1$, i.e., $|Q_I| = |Q_J| = |Q_K| = 1$. But in this case S(Q) is the clique of class 2 associated with $s \in S$ such that the three elements of s are those in Q_I, Q_J and Q_K, respectively. Thus we have

<u>Proof of Theorem 3.3</u>: This is Theorem 6.1 restricted to $p = 1$.||

As mentioned in the proof of Theorem 6.1, the inequality (6.2) can be obtained from the system $Ax = e$ by adding the equations indexed by Q, dividing by 2, then replacing = with ≤ and rounding down each coefficient to the nearest integer. If we now replace ≤ with ≥ and round up rather than down, we obtain a covering type inequality equivalent to (6.2):

<u>Remark 6.2</u>. Let n and Q be as in Theorem 6.1, and let

$$S(Q)_1 := \{s \in S \mid \sum(a_q^s : q \in Q) = 1 \text{ or } 2\}$$

$$S(Q)_2 := \{s \in S \mid \sum(a_q^s : q \in Q) = 3\}$$

Then $x \in P_I$ satisfies (6.2) if and only if it satisfies

$$\sum(x_s : s \in S(Q)_1) + \sum(2x_s : s \in S(Q)_2) \geq p+1. \tag{6.7}$$

<u>Proof</u>: Inequality (6.7) can be obtained from (6.2) by subtracting the equations of $Ax = e$ indexed by Q.||

<u>Proposition 6.3</u>. The number of distinct inequalities (6.2) is $O(2^{3n})$.

<u>Proof</u>: The number of distinct sets Q such that $|Q| = 2p + 1$, $1 \leq p \leq n - 1$ and $1 \leq |Q_L| \leq p$, $L = I, J, K$, is

$$\kappa(Q) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n(i,j)} \binom{n}{i}\binom{n}{j}\binom{n}{k} - 3 \sum_{h=1}^{n-1} \binom{n}{h},$$

where $n(i,j) := \min\{i + j - 1, 2n - 1 - i - j\}$. This is true since for $L = I, J, K$ and $\ell = i, j, k$, there are $\binom{n}{\ell}$ subsets of Q_L of size ℓ, and all values of i, j, k between 1 and $n-1$ can occur, provided that $i + j + k \leq 2n - 1$, $i + j - k \geq 1$, and $\min\{i + j, i + k, j + k\} \geq 3$. The first two of these conditions are ensured by the use of $n(i,j)$ in the summation after k, while the third condition is imposed by subtracting the number of sets in which two of the three indices i, j, k are equal to 1. Further,

$$\kappa(Q) \leq \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \binom{n}{i}\binom{n}{j}\binom{n}{k}$$

$$= \Big(\sum_{i=1}^{n-1} \binom{n}{i}\Big)\Big(\sum_{j=1}^{n-1} \binom{n}{j}\Big)\Big(\sum_{k=1}^{n-1} \binom{n}{k}\Big)$$

$$\leq 2^{3n}$$

since $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.||

<u>Proposition 6.4</u>. Distinct inequalities (6.2) define distinct facets of P_I.

<u>Proof</u>: Let $S(Q_1)$ and $S(Q_2)$ be the supports of two distinct inequalities (6.2) and let $s \in S(Q_1)\backslash S(Q_2)$, $t \in S(Q_2)\backslash S(Q_1)$. Since each inequality (6.2) is facet defining, there exists $x \in P_I^{S(Q_1)}$ such that $x_s = 1$, $x_t = 0$, and $x^* \in P_I^{S(Q_2)}$ such that $x^*_t = 1$, $x^*_s = 0$. but then $P^{S(Q_1)} \neq P_I^{S(Q_2)}$.||

Next we introduce another class of odd hole inequalities, whose left hand side coefficients can be 0, 1 or 2.

<u>Theorem 6.5</u>. Let $Q \subset R$, $|Q| = 2p + 1$, $2 \leq p \leq n - 1$, $1 \leq |Q_L| \leq p$, $L = I, J, K$, and let H be a $(2p + 1)$-hole whose set of links contains Q. Let

$j_* \in Q_J$, $i_* \in Q_I$ and $k_* \in Q_K$ be consecutive links of H (i.e., such that i_* is adjacent in H to both j_* and k_*), and define

$$S_{i_*} = \{(i,j_*,k_*) \mid i \in Q_I \backslash \{i_*\}\},$$

$$T_{i_*} = \{(i_*,j,k) \mid j \in Q_J \backslash \{j_*\},\ k \in K \backslash Q_K \text{ or } k \in Q_K \backslash \{k_*\},\ j \in J \backslash Q_J\}.$$

Then the inequality

$$(6.8) \qquad \sum(2x_s\colon s \in S_{i_*}) + \sum(x_s\colon s \in S(Q) \backslash (S_{i_*} \cup T_{i_*})) \le p$$

defines a facet of P_I.

<u>Proof</u>: The inequality (6.8) can be obtained as follows. Let $s_* := (i_*,j_*,k_*)$. Add up the 2p equations indexed by $Q \backslash \{i_*\}$ and the inequality

$$(6.9) \qquad \sum(x_s\colon s \in C(S_*)) \le 1$$

(i.e., the class 2 clique inequality associated with s_*), divide through by 2 and round down the coefficients on both sides of the resulting inequality to the nearest integer. Since this is a special case of Chvátal's procedure [4], the resulting inequality is valid for P_I.

Now let $Q_I = \{i_1,\ldots,i_r\}$, $Q_J = \{j_1,\ldots,j_s\}$ and $Q_K = \{k_1,\ldots,k_t\}$, and w.l.o.g. assume $p \ge r \ge s \ge t \ge 1$. Let H be the (2p+1)-hole of G_A defined in the proof of Theorem 5.5, and let $s_* = (i_{p-t+1}, j_{p-t}, k_1)$. The inequality (6.8) does not define an improper face of P_I, since it is easy to exhibit a vector $x \in P_I$ that satisfies it with strict inequality.

To show that (6.8) defines a facet of P_I, we will exhibit dim $P_i (=n^3-3n+2)$ affinely independent points of P_I that satisfy (6.7) with equality. Let

$$S^* := S \backslash (S_{i_*} \cup T_{i_*})$$

$$P_I^* := \text{conv}\,\{x \in \{0,1\}^{|S^*|} \mid A^{S^*}x = e\},$$

and

$$S(Q)^* := S(Q) \backslash (S_{i_*} \cup T_{i_*}).$$

Then (i) $\dim P_I^* = \dim P_I - |S_{i_*}| - |T_{i_*}|$

(ii) The inequality

$$(6.10)\qquad \sum(x_s:\ s\varepsilon S(Q)^*) \le p$$

defines a facet of P_I^*.

To see (i), apply the proof of Theorem 3.3 to P_I^* instead of P_I; and to see (ii), apply the proof of Theorem 6.1 to (6.10) instead of (6.2), while making sure in both cases that the triplet used in the definition of λ_i, μ_j, ν_k has no element in Q, and the triplets on which interchanges are performed do not belong to S_{i_*} or T_{i_*}.

Since (6.10) defines a facet of P_I^*, there exists a set of $d^* := \dim P_I^*$ affinely independent points $y^i \in P_I^*$, $i = 1,\dots,d^*$, that satisfy (6.10) with equality. Let $x^i = (y^i,0)$, $i = 1,\dots,d^*$ be the corresponding points of P_I. Clearly, these points satisfy (6.2) with equality. Now for $q = 1,\dots, |S_{i_{p-t+1}}|$ define x^{d^*+q} by

$$x_s^{d^*+q} = \begin{cases} 1 \text{ for } s = |S^*| + q \text{ and for } p-2 \text{ pairwise orthogonal } a^s,\ s \in H, \\ \qquad \text{such that } a^s \bullet a^{|S^*|+q} = 0 \\ 1 \text{ for } s = (i_\alpha, j_\alpha, k_\alpha),\ \alpha = p+2, p+3,\dots,n \\ 0 \text{ otherwise.} \end{cases}$$

Finally, let $q^* := |S_{i_{p-t+1}}|$, and for each $q = 1,\dots, |T_{i_{p-t+1}}|$ define $x^{d^*+q^*+q}$ by

$$x_s^{d^*+q^*+q} = \begin{cases} 1 \text{ for } s = |S^*| + q^* + q \text{ and for } p \text{ pairwise orthogonal } a^s,\ s \in H \\ \qquad \text{such that } a^s \bullet a^{|S^*|+q^*+q} = 0 \\ 1 \text{ for } s = (i_\alpha, j_\alpha, k_\alpha),\ \alpha = p+2, p+3,\dots,n \\ 0 \text{ otherwise} \end{cases}$$

Now let $q^{**} := |T_{i_{p-t+1}}|$. Then the matrix X whose rows are the vectors x^i, $i = 1,\dots,d^* + q^* + q^{**}$, is of the form

$$X = \begin{pmatrix} Y & 0 \\ X_1 & I \end{pmatrix},$$

where Y has as its rows the vectors y^i, $i = 1,\dots,d^*$, I is the identity matrix of order $q^* + q^{**}$, and (X_1, I) has as its rows the vectors x^{d^*+q}, $q = 1,\dots,q^* + q^{**}$. Clearly, X_1 is of full row rank, i.e., of rank $d^* + q^* + q^{**} = \dim P_I^* + |S_{i_*}| + |T_{i_*}| = \dim P_I$.||

By symmetry, one can define S_{j_*}, T_{j_*}, and S_{k_*}, T_{k_*} analogously to S_{i_*} and T_{i_*}, and obtain facet inducing inequalities of the form (6.2) with S_{i_*} and T_{i_*} replaced by S_{j_*} and T_{j_*} or by S_{k_*} and T_{k_*}. Finally, we have

Theorem 6.6. No odd hole inequality valid for P_I can have a left hand side coefficient greater than 2.

Proof: Let $\alpha x \le p$ be one of the inequalities valid for P_I associated with the $(2p+1)$-hole H, and suppose $\alpha_s \ge 3$ for some $s \in S$. Then $s \in H$ and since a^s has only three 1's, A^H has $p - 2$ pairwise orthogonal columns that are also orthogonal to a^s. Let these columns be $a^{t_1},\dots,a^{t_{p-2}}$. Then there exists $x \in P_I$ such that $x_s = a_{t_1} = \dots = x_{t_{p-2}} = 1$. But then $\alpha x = p - 2 + 3 > p$, a contradiction.||

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3-87
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